Bilinear forms

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In this chapter we study finite-dimensional vector spaces over an arbitrary field $\mathbb{F}$ with a bilinear form defined on the space. This is a generalisation of the notion of an inner product space over $\mathbb{R}$.

1 Definition. Matrix representation.

1.1 Bilinear forms

Let V be a vector space over $\mathbb{F}$.

Definition 1: *A bilinear form on V is a map $g : V \times V \to \mathbb{F}$ such that for any u, u', v, v' in V and scalar $a \in \mathbb{F}$ we have*

1. *(linearity in the first variable) $g(u + u', v) = g(u, v) + g(u', v)$ and $g(au, v) = ag(u, v)$;*
2. *(linearity in the second variable) $g(u, v + v') = g(u, v) + g(u, v')$ and $g(u, av) = ag(u, v)$.*

Remark 2: *Equivalently, $g : V \times V \to \mathbb{F}$ is a bilinear form if and only if for all $u \in V$ the map $l_u : V \to V$ defined by $l_u : v \mapsto g(u, v)$ is a linear form on V and for all $v \in V$ the map $r_v : V \to V$ defined by $r_v : u \mapsto g(u, v)$ is a linear form on V.*

Example 3:

1. Let $(V, \langle, \rangle)$ be an inner product space over $\mathbb{R}$. Then $g : V \times V \to \mathbb{R}$ defined by $g(u, v) = \langle u, v\rangle$ is a bilinear form. In particular, the standard dot product in $\mathbb{R}^n$ is a bilinear form. (Note, however, that this is not so in an inner product space over $\mathbb{C}$. The standard dot product in $\mathbb{C}^n$ is not a bilinear form!)
2. The zero form. $\mathbb{F}$ is an arbitrary field and $g : V \times V \to \mathbb{F}$ is defined by $g(u, v) = 0$ for all $u, v \in V$.
3. $V = \mathbb{F}^2_{col}$ and g is the determinant form:
$$g(u, v) = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = x_1y_2 - x_2y_1$$
for $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
(Since the determinant of a matrix is linear in each of its columns when the remaining $n - 1$ columns are fixed, the example can be generalized to $V = \mathbb{F}^n_{col}$ for $n > 2$. Consider an $n \times n$ matrix with all but two columns fixed, then its determinant, considered as a function of the two remaining columns, is bilinear in its two arguments.)
4. $V = \mathbb{R}^4$ and $g(u, v) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ for $u = (x_1, x_2, x_3, x_4)$ and $v = (y_1, y_2, y_3, y_4)$ (this form is called the Lorentz form, and $\mathbb{R}^4$ endowed with this form is called the Minkowski space – an important tool in the special relativity theory).
5. If g is a bilinear form on V and $f : V \to V$ is a linear operator, then $\tilde{g} : V \times V \to \mathbb{F}$ defined by $\tilde{g}(u, v) = g(f(u), v)$ is also bilinear.

1.2 Matrix representation

Similarly to linear operators, bilinear forms can be defined using matrices (after a basis has been fixed).

Definition 4: *Let g be a bilinear form on a space V, and let $\mathcal{B} = (b_1, b_2, \ldots, b_n)$ be a basis of V. Then the matrix G with entries $g_{ij} = g(b_i, b_j)$ is called the matrix of the bilinear form g with respect to the basis $\mathcal{B}$. We will also call G a Gram matrix of g.*

Lemma 5: *Let g be a bilinear form on a finite dimensional vector space V, let G be its Gram matrix with respect to some basis $\mathcal{B}$.*

If $v, w \in V$ are represented in $\mathcal{B}$ as
$[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ *and* $[w]_{\mathcal{B}} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ *then*

$$g(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j g_{ij} = [v]_{\mathcal{B}}^t G [w]_{\mathcal{B}}$$

This shows that the form is defined uniquely by its matrix.

Proof. We have

$$\begin{aligned} g(v, w) &= g(v, \sum_{j=1}^{n} w_j b_j) = \sum_{j=1}^{n} w_j g(v, b_j) \text{ by linearity in the second variable} \\ &= \sum_{j=1}^{n} w_j g(\sum_{i=1}^{n} v_i b_i, b_j) = \sum_{j=1}^{n} \sum_{i=1}^{n} v_i w_j g(b_i, b_j) \text{ by linearity in the first variable} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j g_{ij} \end{aligned}$$

On the other hand we have

$$\begin{aligned} [v]_{\mathcal{B}}^t G [w]_{\mathcal{B}} &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}^t \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}^t \begin{bmatrix} \sum_{j=1}^{n} g_{1j} w_j \\ \sum_{j=1}^{n} g_{2j} w_j \\ \vdots \\ \sum_{j=1}^{n} g_{nj} w_j \end{bmatrix} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j g_{ij} \end{aligned}$$

which proves the claim. □

We see that that the choice of a basis establishes one-to-one correspondence between bilinear forms on an n-dimensional space and $M_n(\mathbb{F})$.

Example 6: 1. The matrix G of the standard inner product of $\mathbb{R}^n$ with respect to its standard basis is the identity matrix $G = I$. More generally, the matrix G of any inner product on $\mathbb{R}^n$ with respect to any of **its** orthogonal bases is $G = I$.

2. The matrix of the zero form with respect to any basis is the zero matrix $G = 0$.

3. The matrix of the determinant form in $V = \mathbb{F}^2_{col}$ with respect to the standard basis of $\mathbb{F}^2_{col}$ is $G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

4. In $\mathbb{R}^4$ the matrix of the Lorentz form $g(u, v) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ with respect to the standard basis is
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

1.3 Change of basis

Just as in the case of linear operators, we would like to know how the matrix of a bilinear form is transformed when the basis is changed.

Here is the calculation: we have seen that for all $v, w \in V$ we have $g(v, w) = [v]^t_{\mathcal{B}} G[w]_{\mathcal{B}}$ where G is the matrix of g with respect to $\mathcal{B}$. In another basis $\mathcal{C}$ this would mean $g(v, w) = [v]^t_{\mathcal{C}} \tilde{G}[w]_{\mathcal{C}}$, where $\tilde{G}$ is the matrix of g with respect to $\mathcal{C}$.

Denote by M the change of basis matrix $M^{\mathcal{C}}_{\mathcal{B}}$, it is invertible and we have $[v]_{\mathcal{B}} = M[v]_{\mathcal{C}}$ and $[w]_{\mathcal{B}} = M[w]_{\mathcal{C}}$. Putting all this together, we get

$$[v]^t_{\mathcal{C}} \tilde{G}[w]_{\mathcal{C}} = g(v, w) = [v]^t_{\mathcal{B}} G[w]_{\mathcal{B}} = (M[v]_{\mathcal{C}})^t GM[w]_{\mathcal{C}} = [v]^t_{\mathcal{C}} (M^tGM)[w]_{\mathcal{C}}$$

The calculation shows that the matrix of g with respect to $\mathcal{C}$ is M^tGM. Hence two matrices $G, \tilde{G}$ represent the same bilinear form with respect to different bases iff there exists an invertible matrix M such that $\tilde{G} = M^tGM$

Remark 7: *Contrast this with the case of matrices representing the same operators: A and A' represent the same operator with respect to two different bases iff there exists an invertible matrix M such that $A' = M^{-1}AM$.*

This prompts the following

Definition 8: *Let $\mathbb{F}$ be a field. Two matrices $A \in M_n(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ are called **congruent** if there exists an* invertible *$P \in M_n(\mathbb{F})$ such that $A = P^tBP$.*

Using the notion of congruence, the result we have shown can be stated in the following way:

Proposition 9: *Two square matrices represent the same bilinear form with respect to two bases if and only if they are congruent.*

2 Symmetric forms. Existence of an orthogonal basis.

In this section we assume that the field $\mathbb{F}$ has characteristic distinct from 2 (i.e. the sum $1 + 1$ is not zero).

Definition 10: *Let g be a bilinear form defined on V.*
Then the bilinear form g^t defined by $g^t(u, v) = g(v, u)$ for all $u, v \in V$ is called the transposed form.
*If $g = g^t$ then the form g is called **symmetric**; if $g = -g^t$ then the form g is called **anti-symmetric**.*

Remark 11: *A form is symmetric iff its matrix with respect to any basis (and hence to every basis) is symmetric, that is iff it satisfies $A^t = A$.*

It is easy to see that in examples 1-4 in Section 1.2 the form is either symmetric or anti-symmetric, with the exception of the zero form, which is the unique form which is both symmetric and anti-symmetric at the same time.

We will be interested in **symmetric** bilinear forms. The theory developed here is a generalization of the theory of inner product real vector spaces.

Indeed, a symmetric bilinear form satisfies the first two axioms of a real inner product, but the third axiom (the positivity condition) has been dropped. Thus it is possible to have for a non-zero vector $v \in V$: $g(v, v) = 0$ or $g(v, v) < 0$.

2.1 Orthogonality

Despite this difference, the orthogonality notation is used without change:

Definition 12: *For $u, v \in V$ we write $u \perp_g v$ and say "u is orthogonal to v with respect to the form g" if $g(u, v) = 0$. The reference to the form can be omitted when g is clear from the context, and the notation can be short-handed to $u \perp v$.*

Because g is symmetric, this is a symmetric notion thus we can write indifferently $u \perp v$ or $v \perp u$.

Example 13:

1. If $g = 0$ then $u \perp v$ for all $u, v \in V$.
2. The vector $v = (1, 0, 0, -1)$ is orthogonal to itself with respect to the Lorentz form in $\mathbb{R}^4$.
3. The vector $e_2 = (0, 1)$ is orthogonal to any vector in $\mathbb{R}^2$ with respect to the form given by $G = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ in the standard basis (e_1, e_2) of $\mathbb{R}^2$.

Remark 14: *Just as we defined the notion of orthogonality for symmetric forms, it can be defined for anti-symmetric forms (and it will still be a symmetric relation: $u \perp v$ if and only if $v \perp u$). In that setting every vector is orthogonal to itself. Indeed, for all $u \in V$ holds $g(u, u) = -g(u, u)$ so $2g(u, u) = 0$, hence $g(u, u) = 0$. (We consider the case where char $\mathbb{F} = 0$ therefore here and in the sequel we are free to divide by 2 or by 4. $2 \neq 0$ so $4 = 2 \cdot 2 \neq 0$.)*

2.2 Quadratic form associated to a bilinear form

Definition 15: *Let g be a symmetric bilinear form on V. The function $q : V \to \mathbb{F}$ defined by $q(v) = g(v, v)$ is called the **quadratic form** associated with g.*

Recall that the term "form" is used for any map that returns a scalar as its output (e.g. a linear form takes a vector and returns a scalar, a bilinear form takes two vectors and retrurns a scalar, ...). The adjective "quadratic" refers to the following property: if $\lambda \in \mathbb{F}$, and $v \in V$, we have

$$q(\lambda v) = g(\lambda v, \lambda v) = \lambda^2 g(v, v) = \lambda^2 q(v)$$

The following proposition shows that no information is lost when a symmetric bilinear form is replaced by its associated quadratic form: g is uniquely defined by q.

Proposition 16: *(Polarization formula for symmetric bilinear forms) Let $q : V \to \mathbb{R}$ be a quadratic form associated to a symmetric bilinear form g. We have*

$$g(u, v) = \frac{q(u + v) - q(u) - q(v)}{2}$$

In particular, no two distinct bilinear forms have the same associated quadratic form.

Proof. Note that for all $u, v \in V$ we have

$$q(u + v) - q(u) - q(v) = g(u + v, u + v) - g(u, u) - g(v, v) = 2g(u, v)$$

which proves the result (here we use the assumption on char$\mathbb{F} \neq 2$ to divide both side of the equation by 2).

Now if $\tilde{g}$ is another symmetric bilinear form whose associated quadratic form is q, then the formula holds for $\tilde{g}$, that is for all $u, v \in V$ we have

$$\tilde{g}(u,v) = \frac{q(u+v) - q(u) - q(v)}{2}$$

Thus we have $g = \tilde{g}$. $\square$

Remark 17: *Alternatively, one can use the polarization identity, whose version appeared in the chapter on inner product spaces. It holds for a general symmetric bilinear form without any change:*

$$g(u,v) = \frac{q(u+v) - q(u-v)}{4}$$

The following proposition shows that in our symmetric case there is always a vector which is not self-orthogonal (unless $g \equiv 0$).

Proposition 18: *For any symmetric bilinear form g on V which is not identically zero there exists a vector $v \in V$ such that $g(v,v) \neq 0$.*

Proof. Let g be a bilinear form which is not identically zero. Use any of the two identities above. There is a pair of vectors $u, v \in V$ for which the left-hand side is non-zero. Hence at least one of the terms in the right-hand side is non-zero, and there exists $w \in \{u, v, u+v\}$ for which $q(w) \neq 0$. $\square$

2.3 Kernel of a bilinear form. Perp.

Definition 19: *The* ***kernel*** *of a bilinear form g is the set $V_0 = \{u \mid g(u,v) = 0 \text{ for all } v \in V\}$.*
A bilinear form g on a space V is called ***non-degenerate*** *if the set is trivial: $V_0 = \{0\}$.*

Equivalently, given a vector u, one can think of the linear form $g(u,\cdot) : V \to \mathbb{F}$ defined by $v \mapsto g(u,v)$. Then g is non degenerate iff for any nonzero vector u, the form $g(u,\cdot)$ is not the zero form (which sends everyone to zero).

Example 20:

1. Inner product is a non-degenerate bilinear form.
2. The Lorentz form is a non-degenerate bilinear form in $\mathbb{R}^4$ (prove this).

If the matrix of g with respect to some basis $\mathcal{B}$ is G, we have

$$\begin{aligned}
u \in V_0 \quad &\iff \quad \text{for any } v \in V, g(u,v) = [u]_{\mathcal{B}}^t G [v]_{\mathcal{B}} = 0 \\
&\iff \quad \text{for } i = 1, \ldots, n \ [u]_{\mathcal{B}}^t G [b_i]_{\mathcal{B}} = 0 \\
&\iff \quad [u]_{\mathcal{B}}^t G = 0 \iff G^t [u]_{\mathcal{B}} = 0
\end{aligned}$$

Proposition 21: *A bilinear form g on a space V is non-degenerate if and only if its matrix G with respect to a basis $\mathcal{B}$ of V is non-degenerate (i.e. non singular).*

Proof. Let $\mathcal{B}$ be a basis of V. Recall that $g(v,w) = [v]_{\mathcal{B}}^t G [w]_{\mathcal{B}}$.
g is degenerate $\iff$ $V_0 \neq 0$ $\iff$ there exists $u \neq 0$ such that $G^t[u]_{\mathcal{B}} = 0$ $\iff$ G^t is degenerate (singular) $\iff$ G is singular (recall that $\det G = \det G^t$). $\square$

Let W be a vector subspace of V. We define its perp, $W^\perp$, just as in inner product spaces.

Definition 22: $W^\perp = \{u \mid g(u,w) = 0 \text{ for all } w \in W\}$

Note that in this notation $V_0 = V^\perp$.

2.4 Orthogonal bases

Definition 23: *A basis $\mathcal{B} = (b_1, \ldots, b_n)$ in V is said to be* **orthogonal** *with respect to g if for all $i \neq j$ we have $g(b_i, b_j) = 0$.*

Remark 24: *In other words, a basis $\mathcal{B}$ is an orthogonal basis for g if and only if the Gram matrix G of g with respect to $\mathcal{B}$ is a* **diagonal** *matrix.*

The following theorem shows that if g is a bilinear symmetric form, then there always exists an orthogonal basis with respect to g.

Theorem 25: *We assume char $\mathbb{F} \neq 2$. Let V be a finite dimensional vector space over $\mathbb{F}$, and let g be a symmetric bilinear form g over V. Then there exists an orthogonal basis for V with respect to g.*

Proof. We prove the claim by induction on $\dim V$. If $\dim V = 1$, then any basis is orthogonal.

Let $\dim V = n$, $n \geq 2$ and assume the claim to be true for any space of dimension smaller than n. Either g is identically zero on V (then any basis is an orthogonal basis for g), or, by Proposition 18, there is a vector $w \in V$ such that $g(w, w) \neq 0$.

Write $W = \mathrm{Span}(w)$. We prove now that $V = W \oplus W^{\perp}$:

1. for any $v \in V$, we write $v = v_1 + v_2$ where $v_1 = \frac{g(w,v)}{g(w,w)} w$ and $v_2 = v - v_1$. Note that $v_1 \in W$, and
$$g(w, v_2) = g(w, v - \frac{g(w,v)}{g(w,w)} w) = g(w, v) - \frac{g(w,v)}{g(w,w)} g(w, w) = 0$$
so $v_2 \in W^{\perp}$. We have proved $V = W + W^{\perp}$.
2. If $v \in W \cap W^{\perp}$ then $v = aw$ for $a \in \mathbb{F}$, and $g(w, aw) = ag(w, w) = 0$ - now $g(w, w) \neq 0$ so we must have $a = 0$ thus $v = 0$. We have proved $W \cap W^{\perp} = \{0\}$.

The restriction $g|_{W^{\perp}}$ is a symmetric bilinear form on a vector space of dimension $< n$, therefore by the induction assumption $W^{\perp}$ has a basis $(v_1, ..., v_{n-1})$ which is orthogonal with respect to g. But any vector in $W^{\perp}$ is orthogonal to w, hence this is true of the v_i's. On the other hand, $V = W \oplus W^{\perp}$ hence the union of the basis (w) for W with the basis $(v_1, ..., v_{n-1})$ for $W^{\perp}$ forms a basis for V.

Then $\mathcal{B} = (w, v_1, ..., v_{n-1})$ is a basis of V which is orthogonal with respect to g. □

Corollary 26: *Any symmetric matrix is congruent to a diagonal one, in other words, if $A \in M_n(\mathbb{F})$ is symmetric, there is an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^t AP$ is a diagonal matrix.*

Corollary 27: *Let $q : V \to \mathbb{F}$ be the quadratic form associated to a symmetric bilinear form g. There exists a basis $\mathcal{B}$ and scalars $a_1, \ldots, a_n$ such that for any vector $v \in V$ if $[v]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have*

$$q(v) = a_1 x_1^2 + \ldots + a_n x_n^n$$

2.5 Diagonalization algorithm

We are looking for an invertible matrix P such that $D = P^t AP$ is diagonal. (If A is the Gram matrix of g in a basis $\mathcal{B}$, then columns of P are the coordinates of vectors of the new orthogonal basis $\mathcal{C}$ in the old basis $\mathcal{B}$: $[v]_{\mathcal{B}} = P[v]_{\mathcal{C}}$, and

$$g(v, w) = [v]^t_{\mathcal{B}} A[w]_{\mathcal{B}} = [v]^t_{\mathcal{C}} [P^t AP][w]_{\mathcal{C}}$$

The matrix P will be built as a product of elementary matrices, corresponding to elementary operations performed on rows of a matrix (rows interchange, multiplication by a non-zero constant, addition to a row a multiple of another one).

Performing such operation on rows (columns) of a matrix A is the same as multiplying A by an invertible elementary matrix on the left (on the right). So if we perform *the same* elementary operation on the rows and then on the columns, we get $Q_1^t A Q_1$, then $Q_2^t Q_1^t A Q_1 Q_2 = (Q_1 Q_2)^t A Q_1 Q_2$, etc, finally arriving at $D = (Q_1 \dots Q_n)^t A (Q_1 \dots Q_n)$.

This way we get both $P = Q_1 \dots Q_n$ and $D = P^t A P$. To keep track of changes of bases, we put the unit matrix on the right of A and simultaneously perform the same operations on the rows only of that matrix to get $(Q_1 \dots Q_n)^t I = P^t$.

(We could get P itself instead of P^t, doing elementary operations on the *columns* of the matrix on the right, but our routine is slightly more convenient – the row operations are performed on the rows of the "big" double-sized matrix, while the column operations are performed on its left half only. A small price to pay is to remember that in the end of the process the new basis should be read from the rows, rather than columns, of the matrix on the right.

Example 28: Perform this diagonalization for

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -3 & 7 & -5 & 0 & 1 & 0 \\ 2 & -5 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 3 & 1 & 0 \\ 0 & 1 & 4 & -2 & 0 & 1 \end{array}\right] \xrightarrow[-2C_1+C_3 \to C_3]{3C_1+C_2 \to C_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 3 & 1 & 0 \\ 0 & 1 & 4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2+2R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 3 & 1 & 0 \\ 0 & 0 & 9 & -1 & 1 & 2 \end{array}\right] \xrightarrow{C_2+2C_3 \to C_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 18 & -1 & 1 & 2 \end{array}\right]$$

So

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 18 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

If A is the matrix of g in the standard basis of $\mathbb{R}^3_{col}$, then $\mathcal{D} = (p_1, p_2, p_3)$ where

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is an orthogonal basis for g.

3 Classification of bilinear forms

We want to understand all the possible symmetric bilinear forms over finite dimensional vector spaces up to a change of basis, in other words, we want to understand what are the possible congruence classes for a mtrix. The answer depends on the firld over which we work, we will only examine the cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$.

3.1 The complex case

Let V be a finite dimensional vector space over $\mathbb{C}$. Let g be a symmetric bilinear form over V. We saw that there exists an orthogonal basis $\mathcal{B} = (b_1, \dots, b_n)$. Now up to reordering $\mathcal{B}$, we can assume that there is an index $k \leq n$ such that $g(b_i, b_i) \neq 0$ iff $i \leq k$.

For each $i \leq k$, we set $\alpha_i \in \mathbb{C}$ to be such that $\alpha_i^2 = g(b_i, b_i)$. We now define a new basis $\mathcal{B}' = (b'_1, \dots, b'_n)$ where $b'_i = \frac{b_i}{\alpha_i}$ for $i \leq k$ and $b'_i = b_i$ for $k < i \leq n$.

The Gram matrix for g in $\mathcal{B}'$ is $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$.

Remark 29: *We have $n-k = \dim V_0$, indeed we saw that $v \in V_0$ iff $G^t[v]_{\mathcal{B}'} = 0$ hence $\dim V_0 = \dim\ker G^t$.*

If two forms are represented by the same matrix up to a change of basis, their kernels must have the same dimension. On the other hand, we just saw that any symmetric bilinear form g there is a basis in which the Gram matrix is of the form $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ where $n-k$ is the dimensuion of the kernel.

Hence there are $n+1$ classes of symmetric bilinear forms over V, one for each possible dimension of the kernel.

3.2 The real case

Now consider the case V be a finite dimensional vector space over $\mathbb{R}$. Let g be a symmetric bilinear form defined on V.

We can do a similar trick as in the complex case, replaceing each b_i such that $g(b_i, b_i) \neq 0$ by $b_i' = b_i/\sqrt{|g(b_i, b_i)|}$. We get that $g(b_i', b_i') \in \{1, -1, 0\}$ for all $i = 1, \dots, n$. Taking vectors of the basis in appropriate order, we can obtain the Gram matrix in the block form

$$G_{p,m,z} = \begin{bmatrix} I_p & & \\ & -I_m & \\ & & 0_z \end{bmatrix}$$

As in the complex case, we will prove that the dimension of the submatrices above depend only on g. Here again, we can see that $v \in V_0$ iff $G^t[v]_{\mathcal{B}} = 0$ iff $[v]_{\mathcal{B}} \in \mathrm{Span}(b_{m+p+1}, \dots, b_n)$. In particular, $z = \dim V_0$.

To understand the geometric meaning of p and m we will need the following definitions.

Definition 30: *Let g be a symmetric bilinear form on a vector space V over $\mathbb{R}$. We say that g is **positive definite** if for any nonzero vector $v \in V$, we have $g(v,v) > 0$. It is **negative definite** if for any nonzero vector $v \in V$, we have $g(v,v) < 0$.*

Example 31: An inner product on a real vector space is exactly a symmetric bilinear positive definite form on V (linearity in the second variable, symmetry, and positive definiteness were the three properties defining inner products over $\mathbb{R}$).

In general however a symmetric bilinear form is neither positive nor negative, but we can look at its restriction to subspaces of V.

Definition 32: *Let g be a symmetric bilinear form. We define:*
$p(g)$ to be the maximum dimension of a subspace W such that $g|_W$ is positive definite,
$m(g)$ be the maximum dimension of a subspace W such that $g|_W$ is negative definite,
$z(g) = \dim V_0$.

Example 33: Let $V = \mathbb{R}^3$, $\mathcal{B}$ the standard basis, and $[g]_{\mathcal{B}} = G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

For $v = (x,y,z) \in \mathbb{R}^3$ we have $g(v,v) = x^2 + y^2 - z^2$. (Thus $q(v) = q(x,y,z) = x^2 + y^2 - z^2$ is the quadratic form defined by g. This example is a 3-dimensional analog of the 4-dimensional Minkowski space with the Lorentz form defined above.)

So $\mathbb{R}^3$ is split into 3 parts – the cone of self-orthogonal vectors $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}$, the set of vectors inside the cone $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 0\}$, and the set of vectors outside the cone $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 > 0\}$. Draw the picture or refer to:
`https://en.wikipedia.org/wiki/Minkowski_space#/media/File:World_line.svg`

Our goal is to find p, m, n for this particular example. We have $n = 0$ because the null space of g is trivial – $V_0 = \{0\}$: while there are non-zero self-orthogonal vectors, there are no non-zero vectors orthogonal to the whole space. This can be easily checked directly (what are $g(e_1, v)$, $g(e_2, v)$, $g(e_3, v)$ for a general $v = (x, y, z)$?), but it is also implied by 21.

The form g is positive on the 2-dimensional subspace $\{z = 0\}$ but is not positive on any 3-dimensional subspace (the only one such subspace being V itself), so $p = 2$.

The form $(-g)$ is positive (one can say "g is negative") on the 1-dimensional subspace $\{x = y = 0\}$ but is not positive on any 2-dimensional subspace, because no plane through the origin of $\mathbb{R}^3$ is wholly contained inside the cone. Therefore $m = 1$.

It should be noted that the spaces of maximal dimension on which g is positive or negative are not unique. A perturbed x, y-plane or the z-axis would do as well. Besides $\{z = 0\}$ the form g is positive on $\{z + 0.1x = 0\}$ or on $\{z + 0.1x - 0.15y = 0\}$ etc. Besides $\{x = y = 0\}$ the form g is negative on $\{x = 0, y = 0.1z\}$ or on $\{x = 0.1z, y = 0.15z\}$ etc.

Theorem 34 (Sylvester's Law of Inertia)**:** *Let g be a symmetric bilinear form defined on V and $\mathcal{B}$ be a basis of V which is orthogonal with respect to g. Then the number of vectors v of the basis $\mathcal{B}$ for which $g(v, v) > 0$ is exactly $p = p(g)$, the number of basis vectors v such that $g(v, v) < 0$ is $m = m(g)$, and the number of basis vectors v such that $g(v, v) = 0$ is exactly $z = z(g)$. In particular, these numbers are uniquely defined by g and do not depend on $\mathcal{B}$.*

Definition 35: *The pair (p, m) is called the signature of the bilinear form g (or of the associated quadratic form q).*

(Some sources define signature as a single integer – the difference $p - m$, instead of the pair (p, m).)

Example 36: The inner product has signature $(n, 0)$, the zero form – $(0, 0)$, the Lorentz form – $(3, 1)$. (Sometimes the Lorentz form is defined as the one of signature $(1, 3)$.)

Proof. Let $\mathcal{B} = (\lfloor_\infty, \dots, \lfloor_\backslash)$ be a basis which is orthogonal with respect to g. Suppose without loss of generality that

$$g(b_i, b_i) \begin{cases} > 0 & \text{if } i \leq p \\ < 0 & \text{if } p < i \leq p + m \\ = 0 & \text{if } p + m < i \leq n \end{cases}$$

We need to show that $p = p(g)$, $m = m(g)$, $z = z(g)$.

Let $W_+ = \mathrm{Span}(b_1, \dots, b_p), W_- = \mathrm{Span}(b_{p+1}, \dots, b_{p+m}), W_0 = \mathrm{Span}(b_{p+m+1}, \dots, b_n)$. Then we have $V = W_+ \oplus W_- \oplus W_0$.

z=z(g): Indeed, if $b_j \in W_0$, then for any $b_i \in \mathcal{B}$ we have that either $i \neq j$, so that by orthogonality of $\mathcal{B}$ we get $g(b_j, b_i) = 0$, or $i = j$ but then since $b_i \in W_0$ we have $g(b_i, b_i) = 0$. Hence $b_j \perp \mathrm{Span}(b_1, \dots, b_n) = V$, in other words $b_j \in V_0$. Thus $W_0 \subseteq V_0$.

Conversely, let $v \in V_0$. Decomposing v as we may, $v = w_+ + w_- + w_0$, with the three summands in the three respective spaces and taking $g(v, w_+)$, we obtain

$$0 = g(v, w_+) = g(w_+ + w_- + w_0, w_+) = g(w_+, w_+) + g(w_-, w_+) + g(w_0, w_+) = g(w_+, w_+) + 0 + 0$$

so $w_+ = 0$. Considering $g(v, w_-)$ we prove in a similar way that $w_- = 0$. Hence $v = w_+ + w_- + w_0 = 0 + 0 + w_0 \in W_0$.We have just shown that $V_0 \subset W_0$, thus establishing that the two spaces are equal, and in particular their dimensions are equal: $z' = z$.

p=p(g): It is an immediate check that g is positive definite on $W_+ = \mathrm{Span}(b_1, \dots, b_p)$: if $u = \sum_{i=1}^k u_i b_i \neq 0$ then $g(u, u) = \sum_{i=1}^p u_i^2 g(b_i, b_i) > 0$. In particular, we get that $p \leq p(g)$ by definition of p.

It is left to show that $p(g) \leq p$. To demonstrate this consider 2 spaces – a space W of maximal dimension $p(g)$ on which g is positive definite, and the space $W_- \oplus W_0$. We claim that they intersect trivially: $W \cap (W_- \oplus W_0) = \{0\}$. Indeed, let $v \in W \cap (W_- \oplus W_0)$. Since $v \in W_- \oplus W_0$ we have $g(v, v) \leq 0$ (check this). Since $v \in W$ we have $g(v, v) \geq 0$. Therefore $g(v, v) = 0$ and, again since

$v \in W$, finally $v = 0$. The trivial intersection of W and $W_- \oplus W_0$ implies that their dimensions add up to at most $\dim V$: $p(g) + (m + z) \leq \dim(V)$. However $p + (m + z) = \dim(V)$, so $p(g) \leq p$. The relation $p = p(g)$ is established.

m=m(g): Repeating the argument for $(-g)$ we can get $m = m(g)$. $\square$

Corollary 37: *A symmetric real matrix is congruent to exactly one matrix of the form*

$$G_{p,m,z} = \begin{bmatrix} I_p & & \\ & -I_m & \\ & & 0_z \end{bmatrix}$$

that is, the sizes of blocks of positive ones, negative ones and zeros are uniquely defined by g.

Using the diagonalization procedure developed in the previous section, we get the signature of a form g, among other things ($(2, 1)$ for the example in the end of the previous section as the diagonal matrix D that we obtained has 2 positive entries and one negative entry on the diagonal). However, if we are interested in the signature only, it is probably easier to work with the quadratic form q associated with the given bilinear form g. If q is reduced to an algebraic sums of squares of new variables obtained from the old ones by *invertible* linear changes of variables, then the signature is (p, m), where p is the number of positive terms and m is the number of negative ones.

Example 38: $q(x, y, z) = x^2 + y^2 + 3z^2 - 2xz - 4yz$, or equivalently g is the bilinear form for which the Gram matrix

$$G = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

in the basis, for which x, y, z are the coordinates. (Note that the non-diagonal entries in the matrix come in pairs, so each one is a half of the corresponding coefficient of the quadratic form.)
Completing squares, we get

$$\begin{aligned} q(x, y, z) &= x^2 + y^2 + 3z^2 - 2xz - 4yz \\ &= (x - z)^2 - z^2 + y^2 + 3z^2 - 4yz \\ &= (x - z)^2 + y^2 - 4yz + 2z^2 \\ &= (x - z)^2 + (y - 2z)^2 - 2z^2 \end{aligned}$$

so the signature of g (or of q) is $(2, 1)$.